

Hyperbolicity of second-order in space systems of evolution equations

Carsten Gundlach

School of Mathematics, University of Southampton, Southampton SO17 1BJ, UK

José M. Martín-García

Instituto de Estructura de la Materia, Centro de Física Miguel A. Catalán, C.S.I.C., Serrano 123, 28006 Madrid, Spain

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A possible definition of strong/symmetric hyperbolicity for a second-order system of evolution equations is that it admits a reduction to first order which is strongly/symmetric hyperbolic. We investigate the general system that admits a reduction to first order and give necessary and sufficient criteria for strong/symmetric hyperbolicity of the reduction in terms of the principal part of the original second-order system. An alternative definition of strong hyperbolicity is based on the existence of a complete set of characteristic variables, and an alternative definition of symmetric hyperbolicity is based on the existence of a conserved (up to lower order terms) energy. Both these definitions are made without any explicit reduction. Finally, strong hyperbolicity can be defined through a pseudo-differential reduction to first order. We prove that both definitions of symmetric hyperbolicity are equivalent and that all three definitions of strong hyperbolicity are equivalent (in three space dimensions). We show how to impose maximally dissipative boundary conditions on any symmetric hyperbolic second order system. We prove that if the second-order system is strongly hyperbolic, any closed constraint evolution system associated with it is also strongly hyperbolic, and that the characteristic variables of the constraint system are derivatives of a subset of the characteristic variables of the main system, with the same speeds.

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I. INTRODUCTION

Research in numerical relativity has recently focused on obtaining a well-posed continuum initial-boundary value problem as a starting point for numerical time evolutions of systems such as a black-hole binary. Well-posedness of an initial-boundary value problem implies that an estimate

$$\|\delta u(\cdot, t)\| \leq F(t) \left(\|\delta u(\cdot, 0)\| + \int_0^t \|\delta g(\cdot, \tau)\| d\tau \right) \quad (1)$$

exists, where $u(x, t)$ is the solution, $u(x, 0)$ the initial data, $g(x, t)$ appropriate free boundary data, δ denotes a linear perturbation and $\|\cdot\|$ stands for appropriate norms (which may involve spatial derivatives), and where $F(t)$ is independent of the initial and boundary data. This means that the solution depends continuously on the initial and boundary data. Hyperbolicity is a property of the evolution equations that can be used as an algebraic criterion for well-posedness. We briefly review several notions of hyperbolicity.

Consider a system of quasilinear evolution equations that is first order in both space and time, or

$$\dot{u} = P^i(u)u_{,i} + S(u), \quad (2)$$

where u is a vector of variables and P^i are square matrices.

Definition 1: The system (2) is called *weakly hyperbolic* if the matrix $P^n \equiv n_i P^i$ has real eigenvalues for any unit vector n_i .

Definition 2: The system (2) is called *strongly hyperbolic* if P^n is diagonalisable with real eigenvalues for any n_i , and the matrix T_n that diagonalises it and its inverse T_n^{-1} depend smoothly on n_i .

Definition 3: The system (2) is called *symmetric hyperbolic* if there exists a Hermitian, positive definite matrix H such that HP^n is Hermitian for any direction n_i and where H does not depend on n_i .

The following properties of strongly and symmetric hyperbolic systems give a more practical meaning to the definitions, and we shall use them later to *define* strong and symmetric hyperbolicity for second-order systems. The key concept for strong hyperbolicity is

Definition 4: A characteristic variable with speed $-\lambda$ in the n_i direction is a linear combination \mathbf{u} of the variables u that obeys

$$\partial_t \mathbf{u} = \lambda \partial_n \mathbf{u} + \dots, \quad (3)$$

where n_i is normalised with respect to some metric, $\partial_n \equiv n^i \partial_i$, and the dots denote derivatives transverse to n_i with respect to the same metric, and lower order terms.

If we write \mathbf{u} as $\mathbf{u} = \bar{u}^\dagger u$, where \bar{u} is a constant vector of coefficients, then

$$\partial_t(\bar{u}^\dagger u) = \bar{u}^\dagger P^n \partial_n u + \dots = \lambda \partial_n(\bar{u}^\dagger u) + \dots \quad (4)$$

if and only if \bar{u}^\dagger is a left eigenvector of P^n or equivalently if \bar{u} is an eigenvector of $P^{n\dagger}$. Characteristic variables \mathbf{u} of the first-order reduction therefore correspond to left eigenvectors \bar{u}^\dagger of P^n . This gives us

Lemma 1: A first-order system is strongly hyperbolic if and only if it admits a complete set of characteristic variables with real speeds that depend smoothly on n_i .

The key concept for symmetric hyperbolicity is that of an energy:

Definition 5: An energy ϵ is a quadratic form in u that is positive definite in the sense that $\epsilon = 0$ if and only if $u = 0$, and which is conserved in the sense that there exists a flux ϕ^i quadratic in u such that

$$\dot{\epsilon} = \phi^i_{,i}. \quad (5)$$

With

$$\epsilon \equiv u^\dagger H u, \quad \phi^i \equiv u^\dagger H P^i u, \quad (6)$$

we have

Lemma 2: A linear first-order system with constant coefficients is symmetric hyperbolic if and only if it admits an energy.

For quasilinear systems this energy is conserved in the approximation where $S(u)$ is neglected and $P^i(u)$ is approximated as constant. (Physically, this corresponds to considering small high-frequency perturbations δu .) When boundaries are present, the time derivative of the energy can be estimated in terms of free boundary data.

Strong hyperbolicity of a first-order system is necessary and sufficient for a well-posed Cauchy problem. The Cauchy problem for a merely weakly hyperbolic system is typically ill-posed in the presence of lower-order terms. Symmetric hyperbolicity implies strong hyperbolicity, and is therefore also sufficient for well-posedness of the Cauchy problem. Furthermore, symmetric hyperbolicity can be used to prove well-posedness of the initial-boundary value problem for a certain class of boundary conditions called maximally dissipative [1].

Hyperbolicity for equations or systems of equations of higher than first order is less well-established. A definition of weak hyperbolicity exists for systems of arbitrary order, but as for first-order systems, it does not guarantee well-posedness [2]. Alternatively, a quasilinear system that is second order in both space and time, or

$$P^{\mu\nu}(u, \partial u) u_{,\mu\nu} + S(\partial u, u) = 0, \quad (7)$$

is called hyperbolic if $P^{\mu\nu}$ is a Lorentzian metric, that is if the principal part of the system is that of a wave equation [3]. Christodoulou has recently generalized the idea introducing the concept of regular hyperbolicity, with less strict positivity requirements on the elliptic block of the principal part [4]. Both can be used as criteria for well-posedness of the Cauchy problem. The Einstein equations are second order, but they fit these definitions of hyperbolicity only when written in harmonic gauge, and so there are no standard definitions of hyperbolicity immediately applicable to forms of the Einstein equations commonly used in numerical relativity.

One possible approach to the well-posedness of a second-order system is to reduce it to first order by introducing auxiliary variables, and to define the second-order system to be strongly hyperbolic or symmetric hyperbolic if the reduction is. An ad-hoc definition along those lines has been used by Sarbach and co-authors [5, 6] to prove well-posedness of the BSSN formulation of the Einstein equations. We formalise this approach in Section III.

Independently, Nagy, Ortiz and Reula [7], following Kreiss and Ortiz [8] have used a pseudo-spectral reduction to define strong hyperbolicity. This method does not appear to generalise to symmetric hyperbolicity, intuitively because Fourier transforms cannot be carried out on a domain with arbitrary boundary. We briefly review this approach in Section V. By casting it in the notation of Section III, show that the two definitions of strong hyperbolicity are equivalent.

As a third alternative, Gundlach and Martín-García [9, 10] define strong and symmetric hyperbolicity directly from the second-order system, by focusing on the existence of characteristic variables in strong hyperbolicity, and of an energy in symmetric hyperbolicity. We review

this approach in Section IV, and show that its definitions of both strong and symmetric hyperbolicity are equivalent to those using a first-order reduction.

Outside the main line of this paper, we analyse in Section VI the well-posedness of the propagation of any constraints that the original second-order system is subject to. In Section VII we apply our results for symmetric hyperbolic systems to mixed symmetric hyperbolic-parabolic systems. Section VIII summarises our results.

II. THE SYSTEM

In this short Section, we establish notation for the class of system that we want to investigate, and clarify the relation between systems that are second order in both time and space, only in space, or only in space and that only in some of the variables.

We begin systems that are first order in time. Formulations of the Einstein equations based on the ADM formulation are naturally first order in time and second order in space. Some cannot even be written in second-order in time form (for example because they have an odd number of variables). With a non-zero shift, the first-order form may also be preferable for numerical simulations [11]. For simplicity, we restrict attention to linear systems with constant coefficients. These can be considered as the linearisation and frozen coefficients approximation of a nonlinear system.

The class of systems that we are interested in are not uniformly second-order: some variables u may appear in the evolution equations without second spatial derivatives. Writing $u = (v, w)$, where w are those variables that appear only with first derivatives, we consider therefore

$$\dot{v} = A_1^{ij} v_{,ij} + A_1^i v_{,i} + A_1 v + A_2^i w_{,i} + A_2 w + a, \quad (8)$$

$$\dot{w} = B_1^{ij} v_{,ij} + B_1^i v_{,i} + B_1 v + B_2^i w_{,i} + B_2 w + b. \quad (9)$$

Here v and w are column vectors (not necessarily of the same length) of variables, the capital letters represent constant matrices of the appropriate dimension, and a, b are forcing functions. We assume that while at least one second derivative of every variable v appears in the equations, the number of variables v has been minimised.

To reduce the system (8,9) to first order, we define the auxiliary variables $d_i \equiv v_{,i}$. By taking a spatial derivative of (8), we find

$$\dot{d}_i = A_1^{jk} v_{,ijk} + A_1^j v_{,ij} + A_1 v_{,i} + A_2^j w_{,ij} + A_2 w_{,i} + a_{,i}. \quad (10)$$

This is of a higher order than we started from, unless A_1^{ij} and A_2^i both vanish. (An example of a second-order evolution equation that cannot be reduced to first order is the heat equation $\dot{u} = u''$.) We have [12]

Lemma 3: *The general second-order in space, first-order in time linear system that can be reduced to first order by*

the introduction of auxiliary variables is of the form

$$\dot{v} = \underline{A_1^{ij} v_{,i}} + A_1 v + \underline{A_2 w} + a, \quad (11)$$

$$\dot{w} = \underline{B_1^{ij} v_{,ij}} + B_1^i v_{,i} + B_1 v + \underline{B_2^i w_{,i}} + B_2 w + b. \quad (12)$$

From now on we refer to this as “the” second-order system. We have underlined the highest derivatives. Without loss of generality we assume from now on that B_1^{ij} is symmetric.

In order to understand how general the system (11-12) is, it is interesting to convert it into second-order in both space and time form. Taking a time derivative of (11) and using (12) to replace \dot{w} , we obtain

$$\ddot{v} = A_2 B_1^{ij} v_{,ij} + A_1^i \dot{v}_{,i} + A_2 B_2^i w_{,i} + A_2 B_2 w + \dots \quad (13)$$

where we have written out all second derivatives and all appearances of w . We can eliminate the remaining appearances of w and $w_{,i}$ in terms of \dot{v} using (11) if and only if the matrices of the system obey

$$\text{rank}(A_2) = \text{rank} \begin{pmatrix} A_2 \\ A_2 B_2^i \end{pmatrix} = \text{rank} \begin{pmatrix} A_2 \\ A_2 B_2 \end{pmatrix}. \quad (14)$$

When A_2 is invertible, which in particular implies equal numbers of v and w variables, these conditions are automatically obeyed. On the other hand, any fully second-order system in a set of variables v can be reduced to the form (11-12) by introducing $\dot{v} \equiv w$. Therefore the class of first-order in time, second-order in space systems (11-12) includes the class of fully second-order systems, but is much bigger.

III. FIRST-ORDER REDUCTION METHOD

A. Parameterised reduction

In reducing (11-12) to first order by defining $d_i \equiv v_{,i}$, we can write each occurrence of $v_{,i}$ also as d_i , or a mixture of the two, and similarly we can write $v_{,ij}$ as $d_{i,j}$ or $d_{j,i}$. To parameterise these ambiguities, we formally add multiples of the auxiliary constraint

$$c_i \equiv d_i - v_{,i} = 0 \quad (15)$$

and its antisymmetrised derivative

$$c_{ij} \equiv d_{[j,i]} = c_{[j,i]} = 0 \quad (16)$$

to all three equations. We could not add $c_{(i,j)}$, or any higher derivatives of the auxiliary constraints, without increasing the order of the system. The general reduction to first order is therefore

$$\dot{v} = A_1^i v_{,i} + A_1 v + A_2 w + a + A_3^{ij} c_{ij} + A_3^i c_i, \quad (17)$$

$$\dot{w} = B_1^{ij} d_{i,j} + B_1^i v_{,i} + B_1 v + B_2^i w_{,i} + B_2 w + b + B_3^{ij} c_{ij} + B_3^i c_i, \quad (18)$$

$$\dot{d}_i = A_1^j d_{i,j} + A_1 v_{,i} + A_2 w_{,i} + a_{,i} + D_i^k c_k + D_i^{jk} c_{jk}. \quad (19)$$

From now on, we shall refer to this system as “the” reduction. We shall refer to the constant matrices A_3^i , A_3^{ij} , B_3^i , B_3^{ij} , D_i^j and D_i^{jk} as the reduction parameters. Without loss of generality, we assume that A_3^{ij} , B_3^{ij} and D_k^{ij} are antisymmetric in i and j . The terms in the second-order system that become principal terms in the reduction are the ones underlined in (11-12). We shall call these the principal part of the second-order system.

B. Auxiliary constraint evolution

The evolution of v and d_i implies an evolution of the auxiliary constraints c_i . This auxiliary constraint system can be written in first-order in space and time form by introducing $c_{ij} \equiv c_{[j,i]}$ (as already defined above) as auxiliary variables. This results in

$$\begin{aligned} \dot{c}_i &= A_1^j c_{i,j} - A_3^j c_{j,i} - A_3^{jk} c_{jk,i} \\ &\quad + D_i^j c_j + D_i^{jk} c_{jk}, \end{aligned} \quad (20)$$

$$\dot{c}_{ij} = A_1^k c_{ij,k} - D_{[i}^k c_{k,|j]} - D_{[i}^{kl} c_{kl,|j]}, \quad (21)$$

As the right-hand side of this system of linear PDEs is homogeneous, $c(x, 0) = 0$ implies $\dot{c}(x, 0) = 0$. Assuming that the coefficients are constant, as we do in this paper, on taking a Fourier transform in x^i we obtain a separate ODE for each wavenumber ω^i , and it follows that $c(x, t) = 0$ is the *unique* solution with $c(x, 0) = 0$.

We have shown that if the auxiliary constraints are zero initially, they remain zero at all times. This allows us to prove well-posedness for the reduction to first order, and then restrict it to the subset of solutions that obey the auxiliary constraints in order to infer well-posedness of the original second-order system.

Note that in order to make this argument, well-posedness of the auxiliary constraint system is not required, as we require only existence and uniqueness of the zero solution, not estimates of any non-zero solution.

C. Definition of hyperbolicity

We now focus on the principal part of the first-order reduction,

$$\partial_t u \simeq P^i \partial_i u, \quad (22)$$

where here and in the following \simeq denotes equality up to lower-order terms and now u stands for (v, w, d_i) . The degree of hyperbolicity of the first-order system depends on the the reduction parameters. The appropriate definitions of hyperbolicity are therefore the following:

Definition 1a: *The second-order system (11-12) is defined to be weakly hyperbolic if and only if it admits at least one reduction to first order (17-19) that is weakly hyperbolic.*

Definition 2a: *The second-order system is defined to be strongly hyperbolic if and only if it admits at least one reduction to first order that is strongly hyperbolic.*

Definition 3a: *The second-order system is defined to be symmetric hyperbolic if and only if it admits at least one reduction to first order that is symmetric hyperbolic.*

In the remainder of this Section we derive necessary and sufficient conditions for these definitions to hold, formulated directly in terms of the principal part of the second-order system, without reference to a reduction.

D. 2+1 split

We introduce a matrix notation in the groups of variables v , w and d . In this notation, (22) is

$$\partial_t \begin{pmatrix} v \\ w \\ d_j \end{pmatrix} \simeq P^i \partial_i \begin{pmatrix} v \\ w \\ d_k \end{pmatrix} \quad (23)$$

where

$$P^i \equiv \begin{pmatrix} A_1^i - A_3^i & 0 & A_3^{ik} \\ B_1^i - B_3^i & B_2^i & B_1^{ik} + B_3^{ik} \\ A_1 \delta_j^i - D_j^i & A_2 \delta_j^i & A_1^i \delta_j^k + D_j^{ik} \end{pmatrix}. \quad (24)$$

We expand the tensor indices i , j and k in (24) into their components in the direction n_i and transverse to it. For this purpose we need a positive definite metric tensor γ_{ij} . This can be chosen to be δ_{ij} , or the physical 3-metric for example in applications to general relativity. We can then define the normal component of a tensor such as $d_n \equiv n^i d_i$ where $n^i \equiv \gamma^{ij} n_j$ and n_i is normalised so that $n_i n_j \gamma^{ij} \equiv 1$, and its transverse part such as $(\delta_i^j - n_i n^j) d_j$, which we denote by d_A . For a reason that will become apparent immediately, we also re-order the rows and columns. We call the resulting matrix \mathcal{P} . It is related to P^n by a unitary transformation that depends on n_i , but it is not the n_i component of any vector of matrices. We have

$$\partial_t \begin{pmatrix} w \\ d_n \\ v \\ d_A \end{pmatrix} \simeq \mathcal{P} \partial_n \begin{pmatrix} w \\ d_n \\ v \\ d_B \end{pmatrix} + \text{transverse derivatives}, \quad (25)$$

where

$$\mathcal{P} \equiv \begin{pmatrix} B_2^n & B_1^{nn} & B_1^n - B_3^n & B_1^{nB} + B_3^{nB} \\ A_2 & A_1^n & A_1 - D_n^n & D_n^{nB} \\ 0 & 0 & A_1^n - A_3^n & A_3^{nB} \\ 0 & 0 & -D_A^n & A_1^n \delta_A^B + D_A^{nB} \end{pmatrix}. \quad (26)$$

We write this in shorthand form as

$$\mathcal{P} \equiv \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{C} \end{pmatrix}, \quad \mathcal{A} \equiv \begin{pmatrix} B_2^n & B_1^{nn} \\ A_2 & A_1^n \end{pmatrix}. \quad (27)$$

The eigenvalues of \mathcal{P} are those of P^n , and one is diagonalisable if and only if the other is. Therefore we can investigate weak and strong hyperbolicity using \mathcal{P} . The

fact that \mathcal{P} has a zero block for all choices of the reduction parameters has several interesting consequences for its eigenvalues and eigenvectors, which respectively determine the weak and strong hyperbolicity properties of the first-order system.

E. Eigenvalues of \mathcal{P} : weak hyperbolicity

The first of these consequences is that the eigenvalues of \mathcal{P} (and hence of P^n) are the union of the eigenvalues of \mathcal{A} and \mathcal{C} , and are independent of \mathcal{B} . We will show in Section III F that we can choose the reduction parameters so that \mathcal{C} has real eigenvalues. This gives us

Lemma 4: *The second-order system is weakly hyperbolic according to Definition 1a if and only if \mathcal{A} has real eigenvalues for all n_i .*

We can further analyse the eigenvalues of \mathcal{A} . If we replace n_i by $-n_i$, then B_2^n and A_1^n change signs, while B_1^{nn} and A_2 do not. This means that if λ is an eigenvalue of \mathcal{A} for some n_i then $-\lambda$ is an eigenvalue of \mathcal{A} for $-n_i$. The eigenvalues must therefore either be $\lambda = O(n_i)$ where O is odd in n_i , or they are paired as $\lambda_{\pm} = O(n_i) \pm E(n_i)$ where O is odd and E is even. As a consequence, if the dimension of \mathcal{A} is odd, at least one eigenvalue must be of the form $\lambda = O(n_i)$, and by continuity it must vanish for some n_i .

F. Eigenvectors of \mathcal{P} : strong hyperbolicity

The reduction is strongly hyperbolic if and only if P^n (or equivalently \mathcal{P}) is diagonalisable. In Appendix A we show that a necessary condition for \mathcal{P} to be diagonalisable is that both \mathcal{A} and \mathcal{C} are diagonalisable. If the sets of eigenvalues of \mathcal{A} and \mathcal{C} are disjoint, this is also sufficient. If they have any eigenvalues in common, additional necessary criteria arise which involve \mathcal{B} .

To state these conditions in a simple form, diagonalise \mathcal{A} and \mathcal{C} simultaneously, so that

$$S^{-1}\mathcal{P}S = \begin{pmatrix} \Lambda_{\mathcal{A}} & \tilde{\mathcal{B}} \\ 0 & \Lambda_{\mathcal{C}} \end{pmatrix} \quad (28)$$

where $\Lambda_{\mathcal{A}}$ and $\Lambda_{\mathcal{C}}$ are diagonal matrices with the eigenvalues of \mathcal{A} and \mathcal{C} respectively in the diagonal. Let all repeated eigenvalues be grouped together in these matrices. Then, for each eigenvalue common to \mathcal{A} and \mathcal{C} , the *corresponding block* of $\tilde{\mathcal{B}}$ must vanish if \mathcal{P} is to be diagonalisable.

The block \mathcal{A} does not contain any reduction parameters, and so is determined by the original second-order system. For a given direction n_i , blocks \mathcal{B} and \mathcal{C} are determined completely by the choice of reduction parameters, but there are not enough reduction parameters to make this true for all directions n_i at once. (For example, as B_1^{ij} is symmetric and B_3^{ij} is antisymmetric in ij , $B_1^{nB} + B_3^{nB}$ can be made to vanish for any one direction n_i , but not for all directions.)

We shall consider one choice of reduction parameters in discussing strong hyperbolicity (here) and another one in discussing symmetric hyperbolicity (in the next subsection). Both choices set

$$\begin{aligned} A_3^{ij} &= 0, \\ A_3^i &= A_1^i, \\ B_3^i &= B_1^i, \\ D_i^j &= A_1 \delta_i^j, \end{aligned} \quad (29)$$

This partial choice has the effect of decoupling v from the w and d_i . To discuss strong hyperbolicity in the case of three space dimensions, we complete the choice of reduction parameters by

$$\begin{aligned} B_3^{ij} &= 0, \\ D_j^{ik} &= \delta_j^i A_1^k - \delta_j^k A_1^i + i\mu \epsilon_j^{ik}, \end{aligned} \quad (30)$$

where μ is a real constant, ϵ_{ijk} is the totally antisymmetric tensor in three dimensions and $i = \sqrt{-1}$. This gives

$$\mathcal{P} = \begin{pmatrix} B_2^n & B_1^{nn} & 0 & B_1^{nB} \\ A_2 & A_1^n & 0 & A_1^B \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mu \epsilon_A^{nB} \end{pmatrix}. \quad (31)$$

The diagonal block $i\mu \epsilon_A^{nB}$ is diagonalisable with real eigenvalues $\pm\mu$. The complexification is unusual, but the term multiplied by $i\mu$ is proportional to the auxiliary constraint c_{ij} , and so has no influence on the original second-order system. If we choose μ large enough, the eigenvalues of this block are distinct from those of the complementary diagonal block (containing \mathcal{A} and a zero row and column), and we have found a choice of reduction parameters that makes \mathcal{P} diagonalisable if \mathcal{A} is diagonalisable. The existence of this choice of reduction parameters completes our proof of

Theorem 1: *The second-order system is strongly hyperbolic according to Definition 2a if and only if \mathcal{A} is diagonalisable for all n_i where the diagonalising matrix and its inverse depend smoothly on n_i .*

Our proof of this theorem assumes three space dimensions, but we suspect that the theorem holds in any number of space dimensions.

Note that this criterion is based only on the coefficients on the second-order system, without explicit reference to the reduction. Note also that for the choice of reduction parameters we have used here, the auxiliary constraint system is strongly hyperbolic, see Appendix B.

G. Symmetric hyperbolicity

According to Definition 3, the reduction is symmetric hyperbolic if and only if there is a Hermitian matrix H such that

$$(HP^n)^\dagger = HP^n \quad (32)$$

for all n_i , with H independent of n_i and positive definite. Note that in the definition (32) we cannot replace P^n by \mathcal{P} .

We again make the partial choice (29) of reduction parameters. B_3^{ij} and D_i^{jk} will be determined in the following in terms of H . The resulting form of P^i is [see (23) for the definition of P^i]

$$P^i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B_2^i & B_1^{ik} + B_3^{ik} \\ 0 & A_2 \delta_j^i & A_1^i \delta_j^k + D_j^{ik} \end{pmatrix}. \quad (33)$$

Clearly it is sufficient to find a symmetriser for the (w, d_i) block. We parameterise H as

$$u^\dagger H u = (v^\dagger, w^\dagger, d_m^\dagger) \begin{pmatrix} 1 & 0 & 0 \\ 0 & K & L^j \\ 0 & L^{\dagger m} & M^{mj} \end{pmatrix} \begin{pmatrix} v \\ w \\ d_j \end{pmatrix} \quad (34)$$

with K and M Hermitian and positive, $K^\dagger = K$, $K > 0$, and $M^{\dagger mj} = M^{jm}$, $M > 0$.

The nontrivial, (w, d_i) block of HP^i is

$$\begin{pmatrix} KB_2^i + L^i A_2 & K(B_1^{ik} + B_3^{ik}) + L^k A_1^i + L^j D_j^{ik} \\ L^{\dagger m} B_2^i + M^{mi} A_2 & L^{\dagger n} (B_1^{in} + B_3^{in}) + M^{mk} A_1^i + M^{mj} D_j^{ik} \end{pmatrix}. \quad (35)$$

A necessary condition for this to be Hermitian is for the matrix

$$\begin{pmatrix} KB_2^n + L^n A_2 & KB_1^{nn} + L^n A_1^n \\ L^{\dagger n} B_2^n + M^{nn} A_2 & L^{\dagger n} B_1^{nn} + M^{nn} A_1^n \end{pmatrix} \quad (36)$$

to be Hermitian for all n_i . This is just the condition that \mathcal{A} admits a symmetriser, or

$$\mathcal{H}\mathcal{A} = (\mathcal{H}\mathcal{A})^\dagger, \quad \text{where} \quad \mathcal{H} \equiv \begin{pmatrix} K & L^n \\ L^{\dagger n} & M^{nn} \end{pmatrix}, \quad (37)$$

for all n_i . If the system is strongly hyperbolic, \mathcal{A} always admits a symmetriser formed from its eigenvectors. Nevertheless, (37) is a non-trivial condition because \mathcal{H} must form a part of H , so that its blocks L^n and M^{nn} are given by $L^i n_i$ and $M^{ij} n_i n_j$, where K , L^i and M^{ij} do not depend on n_i . We now show that (37) actually implies that all of (35) is Hermitian for a particular choice of reduction parameters B_3^{ij} and D_i^{jk} . We do this for each block of (35) in turn.

From the top left block of (35) we have

$$(KB_2^i + L^i A_2)^\dagger = KB_2^i + L^i A_2. \quad (38)$$

This equation does not contain any reduction parameters, and it is clearly equivalent to the condition of $KB_2^n + L^n A_2$ being Hermitian for all n_i , which is contained in the first diagonal block of (37).

The off-diagonal blocks of (35) give

$$(L^{\dagger k} B_2^i + M^{ki} A_2)^\dagger = K(B_1^{ik} + B_3^{ik}) + L^k A_1^i + L^j D_j^{ik}. \quad (39)$$

If we denote this equation by $T^{ik} = 0$, then its symmetric part $T^{(ik)} = 0$, or

$$B_2^{\dagger(i} L^{k)} + A_2^\dagger M^{(ik)} = K B_1^{ik} + L^{(k} A_1^{i)}, \quad (40)$$

again does not contain any reduction parameters. Furthermore $T^{(ik)} = 0$ if and only if $T^{ik} n_i n_k = 0$ for all n_i (this is a property of all totally symmetric tensors), and this is precisely the condition contained in the off-diagonal terms of (37). The antisymmetric part $T^{[ik]} = 0$, or

$$B_2^{\dagger[i} L^{k]} + A_2^\dagger M^{[ik]} = K B_3^{ik} + L^j D_j^{ik}, \quad (41)$$

can be solved for B_3^{ij} because K is by assumption invertible.

Finally, the bottom right block of (35) gives

$$L^{\dagger m} (B_1^{ik} + B_3^{ik}) + M^{mk} A_1^i + M^{mj} D_j^{ik} = [L^{\dagger k} (B_1^{im} + B_3^{im}) + M^{km} A_1^i + M^{kj} D_j^{im}]^\dagger. \quad (42)$$

If we write this as $T^{mik} = 0$, the totally symmetric part $T^{(mik)} = 0$, or

$$L^{\dagger(m} B_1^{ik)} + M^{(mk} A_1^i) = [L^{\dagger(k} B_1^{im}) + M^{(km} A_1^i)]^\dagger. \quad (43)$$

does not contain any reduction parameters. It is equivalent to $T^{mik} n_m n_i n_k = 0$ for all n_i , and so vanishes because of the last diagonal block of (37). After solving (41) for B_3^{ij} we write (42) as

$$U^{mik} + \bar{M}^{mj} D_j^{ik} = U^{\dagger kim} + (\bar{M}^{kj} D_j^{im})^\dagger, \quad (44)$$

where we have defined

$$U^{mik} \equiv L^{\dagger m} B_1^{ik} + M^{mk} A_1^i + L^{\dagger m} K^{-1} (B_2^{\dagger[i} L^{k]} + A_2^\dagger M^{[ik]}), \quad (45)$$

$$\bar{M}^{ij} \equiv M^{ij} - L^{\dagger i} K^{-1} L^j. \quad (46)$$

\bar{M} is Hermitian ($\bar{M}^{\dagger ij} = \bar{M}^{ji}$) and positive definite [because H is positive definite in particular when restricted to vectors $(v, w, d_i) = (0, -K^{-1} L^k d_k, d_i)$], and hence invertible. We define

$$X^{mik} \equiv U^{\dagger kim} - U^{mik}, \quad D^{mik} \equiv \bar{M}^{mj} D_j^{ik}, \quad (47)$$

and so

$$X^{mik} = -X^{\dagger kim}, \quad (48)$$

while $T^{(mik)} = 0$ is equivalent to

$$X^{(mik)} = 0. \quad (49)$$

(44) becomes

$$D^{mik} - D^{\dagger kim} = X^{mik}, \quad (50)$$

which has the general solution

$$6D^{mik} = X^{kmi} + 2X^{kim} + 3X^{ikm} + 4X^{imk} + 5X^{mki} + Y^{mik}, \quad (51)$$

using both (48) and (49). The object Y^{mik} must obey

$$Y^{mik} = Y^{[mik]}, \quad Y^{mik} = -Y^{\dagger mik}, \quad (52)$$

but is otherwise arbitrary. It parameterises the part of the D_j^{ik} that is not determined by H , and can be set to zero.

We have shown that a necessary and sufficient condition for P^i to admit a symmetriser H is for \mathcal{A} to admit a symmetriser \mathcal{H} that depends on n_i in the particular way given in (37). Therefore we have

Theorem 2: *A necessary and sufficient condition for the second-order system to be symmetric hyperbolic according to Definition 3a is that (37) holds for all n_i for some $H > 0$ parameterised by (34).*

Note that positive definiteness of \mathcal{H} does not imply that of H , because positive definiteness of M^{nn} for all n_i does not imply positive definiteness of M . The difference can be expressed in standard terminology as follows: a double quadratic form M^{ij}_{AB} is rank-1 positive if and only if $M^{ij}_{AB}n_in_jm^Am^B > 0$ for all n_i and m^A ; it is rank-2 positive if and only if $M^{ij}_{AB}n_i^An_j^B > 0$ for all n_i^A . Rank-2 positivity implies rank-1 positivity, but they are not equivalent when the indices i, j, A, B belong to spaces of dimension 3 or larger, as shown in the example given in [15]. This suggests it may be useful to introduce an intermediate concept of hyperbolicity based on positivity of \mathcal{H} for all n_i , rather than positivity of H . This already guarantees that \mathcal{A} is diagonalisable, so that the system is strongly hyperbolic. The imposition of rank-1 positivity is also the key ingredient of the definition of “regular hyperbolicity” by Christodoulou [4] for second order in both space and time systems, which is also known to yield well-posed problems. We believe there is a connection between regular hyperbolicity and our condition $\mathcal{H} > 0$, but have not been able to show it.

IV. DIRECT SECOND-ORDER METHOD

A. Strong hyperbolicity

Following [10], we now elevate Lemma 1 to a definition of strong hyperbolicity for second-order in space, first-order in time systems, with the only difference that the \mathbf{u} become linear combinations of $v_{,i}$ and w :

Definition 2b: *The system (11-12) is called strongly hyperbolic if and only if there is a set of characteristic variables \mathbf{u} linear in w and $v_{,i}$ obeying (3) with real speeds λ and where the matrix relating u and \mathbf{u} and its inverse depend smoothly on n_i .*

This definition has two interesting consequences. The first is that $v_{,A}$ can always be considered as a zero speed

variable in the direction n_i because

$$\partial_t(v_{,A}) = \partial_A(\partial_t v) \quad (53)$$

so that the right-hand side can be considered as a sum of transverse derivatives only. (In fact arbitrary linear combinations of the $v_{,A}$ can be given arbitrary speeds.)

The second consequence is that nontrivial characteristic variables are unique only up to adding transverse derivatives. If \mathbf{u} is a characteristic variable with speed $-\lambda$ then, for any vector X^A (made from the u)

$$\begin{aligned} & \partial_t(\mathbf{u} + \partial_A X^A) \\ & \simeq \partial_t \mathbf{u} + \partial_A(\partial_t X^A) \\ & \simeq \lambda \partial_n \mathbf{u} + \text{transv. deriv.} \\ & \simeq \lambda \partial_n \mathbf{u} + \lambda \partial_A(\partial_n X^A) + \text{transv. deriv.} \\ & \simeq \lambda \partial_n(\mathbf{u} + \partial_A X^A) + \text{transv. deriv.} \end{aligned} \quad (54)$$

and so $\mathbf{u} + \partial_A X^A$ is also a characteristic variable with speed $-\lambda$. Such calculations rely on commuting partial derivatives to interpret $\partial_A \partial_n \dots$ either as a normal or a transverse derivative, depending on the situation.

The second-order system has no reduction parameters. However, for the purpose of comparing the second-order approach with the reduction approach, we can translate the second-order approach into the *notation* of a first-order reduction. We account for the fact that $v_{,i}$ and d_i , and $v_{,ij}$ and $v_{,ji}$, are now identical, by allowing the reduction parameters to depend explicitly on n_i (so that they are not tensors.) This allows us to set the blocks \mathcal{B} and \mathcal{C} of \mathcal{P} to arbitrary values for every n_i , and this allows us to make arbitrary linear combinations of $v_{,A}$ and v characteristic variables with arbitrary speeds, and to add arbitrary combinations of $v_{,A}$ and v to any characteristic variables made from w and $v_{,n}$, as we have discussed above. \mathcal{A} being diagonalisable is a necessary condition for \mathcal{P} to be diagonalisable, and with $\mathcal{B} = 0$ and $\mathcal{C} = 0$ it is also sufficient. Therefore Definition 2b is equivalent to \mathcal{A} being diagonalisable and by Theorem 1 it is then equivalent to Definition 2a.

Alternatively, we can write down the principal part of the second-order system in a 2+1 split as follows:

$$\dot{v} \simeq 0, \quad (55)$$

$$\dot{v}_{,n} \simeq A_1^j(v_{,n})_{,j} + A_2 w_{,n}, \quad (56)$$

$$\dot{v}_{,A} \simeq A_1^j(v_{,j})_{,A} + A_2 w_{,A}, \quad (57)$$

$$\begin{aligned} \dot{w} \simeq & B_1^{nn}(v_{,n})_{,n} + 2B_1^{nA}(v_{,n})_{,A} \\ & + B_1^{AB}(v_{,A})_{,B} + B_2^i w_{,i}, \end{aligned} \quad (58)$$

where $v_{,i}$ on the right-hand side is now considered a lower order term. Note the way the second derivatives have been ordered differently in $\dot{v}_{,n}$ and $\dot{v}_{,A}$. In the language of reduction this corresponds to the reduction parameters

depending explicitly on n_i . The matrix \mathcal{P} becomes

$$\partial_t \begin{pmatrix} w \\ v, n \\ v \end{pmatrix} \simeq \mathcal{P} \partial_n \begin{pmatrix} w \\ v, n \\ v \end{pmatrix} + \text{transverse derivatives}, \quad (59)$$

where

$$\mathcal{P} \equiv \begin{pmatrix} B_2^n & B_1^{nn} & 0 & 0 \\ A_2 & A_1^n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (60)$$

Once again \mathcal{P} is diagonalisable if and only if \mathcal{A} is. Either way, and taking into account the smoothness conditions, we have

Theorem 3: *Definition 2b is equivalent to Definition 2a.*

B. Symmetric hyperbolicity

Following [10], we now elevate Lemma 2 to a definition of symmetric hyperbolicity for second-order systems in **Definition 3b:** *The second-order system is called symmetric hyperbolic if and only if it admits a positive definite energy ϵ and a flux ϕ^i , both quadratic in w and $v_{,i}$, that obey*

$$\dot{\epsilon} \simeq \phi^i_{,i}. \quad (61)$$

Note that in obtaining (61) one can make arbitrary use of $v_{,ij} = v_{,ji}$.

We parameterise the energy ϵ by (34), with d_i replaced by $v_{,i}$, and parameterise the flux ϕ^i as

$$\phi^i = (w^\dagger, v_{,m}^\dagger) \begin{pmatrix} F^i & F^{ik} \\ F^{\dagger im} & F^{mik} \end{pmatrix} \begin{pmatrix} w \\ v_{,k} \end{pmatrix}, \quad (62)$$

with $F^{\dagger i} = F^i$ and $F^{\dagger kim} = F^{mik}$. In the second-order system, there are no reduction parameters, and so the non-trivial part of HP^i is given by

$$\begin{pmatrix} KB_2^i + L^i A_2 & KB_1^{ik} + L^k A_1^i \\ L^{\dagger m} B_2^i + M^{mi} A_2 & L^{\dagger m} B_1^{ik} + M^{mk} A_1^i \end{pmatrix}. \quad (63)$$

In a first-order system, energy conservation is precisely equivalent to HP^i being Hermitian. In the second-order system, the relation between HP^i and energy conservation is more complicated, because $v_{,ik} = v_{,ki}$.

To see this more clearly, we write out

$$\begin{aligned} \dot{\epsilon} = & 2 \left[w^\dagger (KB_2^i + L^i A_2) w_{,i} \right. \\ & + w^\dagger (KB_1^{ik} + L^k A_1^i) v_{,ik} \\ & + v_{,m}^\dagger (L^{\dagger m} B_2^i + M^{mi} A_2) w_i \\ & \left. + v_{,m}^\dagger (L^{\dagger m} B_1^{ik} + M^{mk} A_1^i) v_{,ik} \right], \end{aligned} \quad (64)$$

$$\begin{aligned} \phi^i_{,i} = & 2 \left[w^\dagger F^i w_{,i} + w^\dagger F^{(ik)} v_{,ik} \right. \\ & \left. + v_{,m}^\dagger F^{\dagger im} w_i + v_{,m}^\dagger F^{m(ik)} v_{,ik} \right]. \end{aligned} \quad (65)$$

Comparing the first terms in $\dot{\epsilon}$ and $\phi^i_{,i}$ we have

$$F^i = KB_2^i + L^i A_2, \quad (66)$$

$$F^i = F^{\dagger i}, \quad (67)$$

which admits a solution F^i if and only if (38) is obeyed. Comparing the second and third terms we have

$$F^{(ik)} = KB_1^{ik} + L^{(k} A_1^{i)}, \quad (68)$$

$$F^{\dagger im} = L^{\dagger m} B_2^i + M^{mi} A_2, \quad (69)$$

which admits a solution F^{ik} if and only if (40) is obeyed. Comparing the fourth terms we have

$$F^{m(ik)} = L^{\dagger m} B_1^{ik} + M^{m(k} A_1^{i)} \equiv S^{mik}, \quad (70)$$

$$F^{\dagger kim} = F^{mik}. \quad (71)$$

These admit a solution F^{mik} if and only if (43) is obeyed. It is clear that (43) is necessary. To demonstrate that it is sufficient we solve explicitly for F^{mik} . The general solution of (70) is

$$F^{mik} = S^{mik} + J^{mik}, \quad (72)$$

where $J^{mik} = J^{m[ik]}$. Defining

$$V^{mik} \equiv S^{\dagger kim} - S^{mik}, \quad (73)$$

the general solution of (71) is

$$\begin{aligned} 6J^{mik} = & V^{kmi} + 2V^{kim} + 3V^{ikm} \\ & + 4V^{imk} + 5V^{mik} + W^{mik}, \end{aligned} \quad (74)$$

if and only if $V^{(mik)} = 0$, which is equivalent to (43). The remaining free coefficient $W^{mik} \equiv W^{[mik]} \equiv W^{\dagger mik}$ parameterises terms in ϕ^i whose divergence vanishes identically. It can be set equal to zero without loss of generality.

We have shown that the second-order system admits a conserved energy if and only if (38), (40) and (43) hold, which together are equivalent to (37). This is equivalent to the existence of a conserved energy for the first-order reduction. The two energies are in fact the same under the (unambiguous) identification of d_i with $v_{,i}$. This means that the two definitions of symmetric hyperbolicity are equivalent, as both are equivalent to the matrix H defined by (34) being positive definite, and obeying (37) for all n_i . We have

Theorem 4: *Definition 3b is equivalent to Definition 3a.*

Given an energy H of the second-order system, a first-order reduction that admits the same energy is given by the reduction parameters (29) and B_3^{ij} and D_j^{ik} determined in Section III G. Going the other way, an energy H for any first-order reduction is clearly also an energy for the original second-order system. If the second-order system admits a multi-parameter family of energies, then some of these parameters define reduction parameters B_3^{ij} and D_j^{ik} of the first-order system, and the remainder parameterise the energy of that particular reduction. An example of this split is given in Appendix D.

C. Maximally dissipative boundary conditions

A first-order symmetric hyperbolic system on a domain Ω admits an energy

$$E = \int_{\Omega} \epsilon dV \quad (75)$$

whose time derivative is given by the flux through the boundary,

$$\dot{E} \simeq \int_{\partial\Omega} \phi^n dS, \quad (76)$$

where n_i is now the outward pointing normal to the boundary $\partial\Omega$. In Appendix C we show that symmetric hyperbolicity implies strong hyperbolicity and that the boundary flux can be written as

$$\phi^n = \sum_{\alpha} \lambda_{\alpha} \mathbf{u}_{\alpha}^2. \quad (77)$$

where the sum is over a suitable basis of characteristic variables. Therefore the growth of the energy can be controlled by controlling all characteristic variables that are ingoing at the boundary ($\lambda_{\alpha} > 0$) (“maximally dissipative boundary conditions”), while ingoing or tangential characteristic variables give negative or zero contributions to ϕ^n and hence \dot{E} .

The same result holds for a second-order system but, as we have seen, the characteristic variables of the second-order system are not unique: the $v_{,A}$ can be given arbitrary speeds, and arbitrary combinations of $v_{,A}$ can be added to any characteristic variable. In order to impose maximally dissipative boundary conditions with the desired effect of controlling an energy, we need to control all incoming characteristic variables of an actual first-order reduction. We can do this within the second-order system by imposing boundary conditions on characteristic variables \mathbf{u}_{α} constructed from w and $v_{,i}$, but we need to have the correct admixtures of $v_{,A}$ in these characteristic variables.

We have seen that in order to show strong hyperbolicity for the second-order system one needs to diagonalise only the matrix \mathcal{A} . Let us call the characteristic variables of the second-order system given by eigenvectors of \mathcal{A}^{\dagger} , and which are therefore constructed only from w and $v_{,n}$, the “short” characteristic variables, and let us denote them by \mathbf{u}'_{α} . For proving symmetric hyperbolicity one only needs to find a conserved energy $\epsilon(w, v_{,i})$.

Assuming that we already have the \mathbf{u}'_{α} on the one hand, and the energy $\epsilon(w, v_{,i})$ on the other, the simplest way of generating the required “full” characteristic variables \mathbf{u}_{α} is to make the ansatz $\mathbf{u}_{\alpha} = \text{const}(\mathbf{u}'_{\alpha} + \text{undetermined multiples of the } d_A)$, and to determine the coefficients and overall normalisation for each \mathbf{u}_{α} so that the full characteristic variables obey

$$\epsilon = \sum_{\alpha} \mathbf{u}_{\alpha}^2. \quad (78)$$

(76) then follows.

V. PSEUDO-DIFFERENTIAL REDUCTION METHOD

For the purpose of comparison, we now describe the pseudo-differential reduction method of [7, 8] in our notation. We carry out a Fourier transform in space with wave number ω_i of the second-order system (11-12). We denote the Fourier transforms of v and w by \hat{v} and \hat{w} . We choose the direction n_i to be that of the wavenumber ω_i and write $\omega_i \equiv |\omega|n_i$. We then have $\hat{d}_n = i|\omega|\hat{v}$ and $\hat{d}_A = 0$. We can then use \hat{d}_n to represent \hat{v} . The principal part of the Fourier-transformed system can be written as

$$\partial_t \begin{pmatrix} \hat{w} \\ \hat{d}_n \end{pmatrix} \simeq i|\omega|\mathcal{A} \begin{pmatrix} \hat{w} \\ \hat{d}_n \end{pmatrix} \quad (79)$$

where \mathcal{A} is the matrix defined above, and the non-principal terms not shown here are homogeneous of order zero in $|\omega|$. In our notation, the definition of [8] is then

Definition 2c: *The second-order system is called strongly hyperbolic if and only if there exists a pseudo-differential reduction to first order of the form (79) where \mathcal{A} is diagonalisable with real eigenvalues, and the diagonalising matrix and its inverse depend smoothly on n_i .*

We have set up our notation so that from Theorem 1 we immediately have

Lemma 5: *Definition 2c is equivalent to Definition 2a and to Definition 2b.*

As the pseudo-differential approach relies in an essential way on Fourier transforms, it does not lend itself to defining a locally conserved energy. Therefore there is no definition of symmetric hyperbolicity in this approach.

VI. THE EVOLUTION OF CONSTRAINTS ON THE ORIGINAL SECOND-ORDER SYSTEM

In many applications, the original second-order system is subject to differential constraints, which are conserved under evolution. We shall call these the “original constraints” to distinguish them from the “auxiliary constraints” $d_i - v_{,i} = 0$ that arise additionally in the process of first-order reduction.

Note that until now we have not mentioned or used original constraints. The reason is that in general one wants to prove well-posedness of the second-order system *if the original constraints are obeyed or not*. This is important for example if one wants to carry out numerical simulations using “free evolution” where the original constraints are imposed only on the initial data. In the continuum the constraints then remain zero, but in numerical free evolution they are violated through finite differencing error. At a later time one is effectively evolving initial data that do not obey the constraints. Therefore the continuum problem must be well-posed for non-vanishing original constraints as a necessary condition for numerical stability. One may of course use the original

constraints to modify the original, second-order system, but here we assume that this has already been done, and the second-order system is fixed.

We now prove that the evolution system of original constraints is strongly hyperbolic if it closes and if the second-order main system is strongly hyperbolic. An equivalent result for first-order systems subject to constraints is given in [13].

We consider a vector c of constraints which are quasi-linear of the form

$$c \simeq C_1^{ij} v_{,ij} + C_2^i w_{,i}, \quad (80)$$

where the matrix C^{ij} is symmetric in ij . It is clear from (11-12) that the evolution of these constraints is first order in space and time. If the constraint system is closed, its principal part must then be of the form

$$\dot{c} \simeq G^i c_{,i} \quad (81)$$

for a vector of square matrices G^i . Using the second-order evolution equations (11-12) and comparing the leading order terms in v and w we find

$$C_1^{(ij)} A_1^{(k)} + C_2^{(i)} B_1^{(jk)} = G^{(i)} C_1^{(jk)}, \quad (82)$$

$$C_1^{(ij)} A_2 + C_2^{(i)} B_2^{(j)} = G^{(i)} C_2^{(j)}. \quad (83)$$

These identities between totally symmetric matrices hold if and only if their contraction with n_i on all indices hold for all n_i . Writing the nnn and nn components of these equations in matrix form we have

$$(C_2^n, C_1^{nn}) \begin{pmatrix} B_2^n & B_1^{nn} \\ A_2 & A_1^n \end{pmatrix} = G^n (C_2^n, C_1^{nn}) \quad (84)$$

for all n_i . We write this in compact notation as

$$CA = GC. \quad (85)$$

If the second-order main system is strongly hyperbolic, \mathcal{A} is diagonalisable with $\mathcal{A} = T\Lambda T^{-1}$. G can always be brought into Jordan form as $G = SJS^{-1}$. Then

$$\tilde{C}\Lambda = J\tilde{C}, \quad \tilde{C} \equiv S^{-1}CT. \quad (86)$$

We assume that the rows of C , and therefore the rows of \tilde{C} , are linearly independent. This means that there is no redundancy between the differential constraints, and is similar to an assumption in [13].

Consider now the first Jordan block of J with eigenvalue μ_1 . For simplicity assume it has size 2. Exceptionally writing out the internal matrix indices, we have

$$\begin{aligned} \tilde{C}_{1\alpha} \lambda_\alpha &= \mu_1 \tilde{C}_{1\alpha} + \tilde{C}_{2\alpha}, \\ \tilde{C}_{2\alpha} \lambda_\alpha &= \mu_1 \tilde{C}_{2\alpha} \end{aligned} \quad (87)$$

(no sum over the index α). From the second equation $\tilde{C}_{2\alpha} = 0$ for all α such that $\lambda_\alpha \neq \mu_1$. Using this result and the first equation, $\tilde{C}_{2\alpha} = 0$ precisely for those α for which $\lambda_\alpha = \mu_1$. By assumption, no row of \tilde{C} vanishes,

so $\tilde{C}_{2\alpha}$ cannot all vanish. Therefore, there must be at least one α such that $\lambda_\alpha = \mu_1$, and the first equation must be absent, that is, the Jordan block has only size 1. Repeating this argument for all Jordan blocks of J means that each eigenvalue of G coincides with one of \mathcal{A} , and that J is diagonal, that is, G is diagonalisable. Writing $G = S\Lambda'S^{-1}$ where the diagonal matrix Λ' is a submatrix of Λ , we have

$$(SC)\mathcal{A} = \Lambda'(SC). \quad (88)$$

This means that the rows of SC are left eigenvectors of \mathcal{A} , and parameterise to characteristic variables of the second-order system. We have shown

Theorem 5: *The evolution of the original constraints is strongly hyperbolic if the second-order main system is, and its characteristic speeds are then a subset of those of the main system. Furthermore, we can find a basis of characteristic variables for the main system and the constraint system such that for each characteristic variable \mathbf{c}_α of the constraint system, there is a characteristic variable \mathbf{u}_α of the main system such that*

$$\mathbf{c}_\alpha = \partial_n \mathbf{u}_\alpha + \text{transverse derivatives}. \quad (89)$$

Note that there is no such result for symmetric hyperbolicity.

VII. SYMMETRIC HYPERBOLIC-PARABOLIC SYSTEMS

Theorem 4.6.2 of [1] asserts the following: Assume we have a vector of variables u and another vector of variables z , which obey a linear system of evolution equations of the form

$$\partial_t u = D_{11}u + D_{12}z, \quad (90)$$

$$\partial_t z = D_{21}u + D_{22}z. \quad (91)$$

Here the D are linear spatial derivative operators whose coefficients can depend on t and x^i . D_{11} is a first-order derivative operator such that $\partial_t u = D_{11}u$ is symmetric hyperbolic. D_{22} is a second-order derivative operator such that $\partial_t z = D_{22}z$ is parabolic. D_{12} and D_{21} are arbitrary first-order derivative operators. Then the coupled system is called mixed symmetric hyperbolic/parabolic. Its Cauchy problem with periodic boundaries is well-posed.

The theorem can be applied straightforwardly to second-order systems. We identify the variables u of the theorem with the variables (v, w, d_i) of the first-order reduction of what is to be the symmetric hyperbolic subsystem, and then go back to the second-order form of this subsystem by replacing d_i with $v_{,i}$. The result is the system

$$\begin{aligned} \dot{v} &= A_1^i v_{,i} + A_1 v + A_2 w + a + \underline{C}z, \\ \dot{w} &= B_1^{ij} v_{,ij} + B_1^i v_{,i} + B_1 v + B_2^i w_{,i} + B_2 w + b \end{aligned} \quad (92)$$

$$+ \underline{D^i z}_{,i} + Dz, \quad (93)$$

$$\dot{z} = D_{22}z + \underline{E_1^{ij} v}_{,ij} + E_1^i v_{,i} + E_1 v + \underline{E_2^i w}_{,i} + E_2 w. \quad (94)$$

The coupling operators D_{12} and D_{21} are parameterised by the matrices C and D , and E , respectively. We have underlined their principal parts to show what order of derivative is allowed in the coupling terms.

Definition 6: A second-order system is called *mixed symmetric hyperbolic-parabolic* if it is of the form (92-94), such that D_{22} is parabolic and the system (11-12) with the same coefficients is symmetric hyperbolic (in the sense of Definition 2a or 2b).

Theorem 4.6.2 of [1] then gives us

Lemma 6: The Cauchy problem with periodic boundary conditions for such a system is well-posed.

VIII. CONCLUSIONS

We have formalised the definition of strong or symmetric hyperbolicity of a system of evolution equations that are first order in time and second order in space by reducing them to an equivalent first-order system. We have given necessary and sufficient criteria for the existence of a reduction that is strongly hyperbolic or symmetric hyperbolic. These criteria are formulated entirely in terms of the principal part of the second-order system, without an explicit reference to the reduction.

We have proved that the definitions of strong hyperbolicity based on a first-order reduction, a pseudo-differential reduction, and a direct second-order approach are all equivalent. The definitions of symmetric hyperbolicity based on a first-order reduction and on a direct second-order approach are also equivalent.

In order to analyse the well-posedness of a given second-order system in practice, there are three non-trivial calculations to be carried out, independently of the approach in which one has defined hyperbolicity. Suppressing technical details, they are as follows.

Strong hyperbolicity Strong hyperbolicity of the second-order system is equivalent to diagonalisability, with real eigenvalues, of the matrix \mathcal{A} .

Symmetric hyperbolicity Symmetric hyperbolicity of the second-order system is equivalent to the existence of an energy and flux quadratic in the $v_{,i}$ and w .

Maximally dissipative boundary conditions In order to impose maximally dissipative boundary conditions, one needs the full characteristic variables of a symmetric hyperbolic first-order reduction. This is done most easily starting from the left eigenvectors of \mathcal{A} and the energy $\epsilon(w, v_{,i})$.

We have established criteria for well-posedness of the second-order system regardless of any constraints it is subject to. However, if the second-order system is strongly hyperbolic, and there is a closed constraint system associated with it, then the constraint system is also strongly hyperbolic, and the characteristic variables of

the constraint system are related to a subset of the characteristic variables of the main system.

It is known that a first-order symmetric hyperbolic system coupled to a parabolic system through at most first derivatives of all variables has a well-posed Cauchy problem. We have generalised this result to second-order symmetric hyperbolic systems through a reduction to first order.

Appendix D gives an example of a simple second order system discussed in both the first order reduction approach and the direct second-order approach.

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APPENDIX A: DIAGONALISABILITY OF MATRICES WITH A ZERO BLOCK

Consider the matrix

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad (A1)$$

where the square block A has size n and the square block C has size m . We now prove that if M is diagonalisable then A and C are both diagonalisable.

The eigenvalues of M are clearly the union of those of A and C . The eigenvectors of M can be constructed from those of A and C as follows: Suppose we have

$$Av_i = \lambda_i v_i, \quad i = 1, \dots, n, \quad (A2)$$

$$Cw_j = \mu_j w_j, \quad j = 1, \dots, m. \quad (A3)$$

Then we can form the eigenvectors $x_i = (v_i, 0)$, which span the invariant subspace of M , and $y_j = (u_j, w_j)$ such that

$$Mx_i = \lambda_i x_i, \quad i = 1, \dots, n, \quad (A4)$$

$$My_j = \mu_j y_j, \quad j = 1, \dots, m, \quad (A5)$$

with the condition $(A - \mu_j)u_j = -Bw_j$ for $j = 1, \dots, m$. If μ_j is not an eigenvalue of A then $(A - \mu_j)$ can be inverted and there is a unique solution for u_j . Therefore, if the eigenvalues of A and C are disjoint, the eigenvectors can be completed and the x_i are linearly independent from the y_j because they correspond to different eigenvalues. In this case M is diagonalisable if and only if both A and C are diagonalisable.

Now suppose that A and C share an eigenvalue λ . The transformation

$$UMU^{-1} = \begin{pmatrix} U_1AU_1^{-1} & D \\ 0 & U_2CU_2^{-1} \end{pmatrix} \quad (\text{A6})$$

with

$$U = \begin{pmatrix} U_1 & U_1X \\ 0 & U_2 \end{pmatrix} \quad U^{-1} = \begin{pmatrix} U_1^{-1} & -XU_2^{-1} \\ 0 & U_2^{-1} \end{pmatrix} \quad (\text{A7})$$

and $D = U_1(XC - AX + B)U_2^{-1}$ brings A and C into Jordan form simultaneously for suitable U_1 and U_2 . Without loss of generality we can assume that A is a single Jordan block of eigenvalue λ and $\text{rank}(A - \lambda I) = r$, and that C is another Jordan block with the same eigenvalue and $\text{rank}(C - \lambda I) = s$. The matrix M is diagonalisable if $\text{rank}(M - \lambda I) = 0$. We have $\text{rank}(M - \lambda I) \geq r + s$ because the $r + s$ columns of $(M - \lambda I)$ containing a 1 in the second diagonal are linearly independent, while the matrix B could provide additional linearly independent vectors. Therefore if M is diagonalisable we must have $r = s = 0$ for each Jordan block of A and C , and so A and C are diagonalisable.

APPENDIX B: HYPERBOLICITY OF THE AUXILIARY CONSTRAINT SYSTEM

With the choice of reduction parameters that we have used in the proof of Theorem 1, namely (29), (30), and using the following further auxiliary constraints,

$$C_{ij} = c_{ij} + c_{[i,j]} = 0, \quad (\text{B1})$$

$$C_{ijk} = c_{ij,k} + c_{jk,i} + c_{ki,j} = 0, \quad (\text{B2})$$

the auxiliary constraint system can be reduced to the decoupled system

$$\dot{c}_i = A_1 c_i - i\mu \epsilon_i^{jk} c_{j,k}, \quad (\text{B3})$$

$$\dot{c}_{ij} = A_1 c_{ij} - i\mu \epsilon_{[i}^{kl} c_{kl,|j]}. \quad (\text{B4})$$

This is strongly hyperbolic with speeds $0, \pm\mu$ for all n_i . Furthermore

$$\dot{C}_i = A_1 C_i - i\mu \epsilon_i^{jk} C_{j,k}, \quad (\text{B5})$$

$$\dot{C}_{ijk} = A_1 C_{ijk}, \quad (\text{B6})$$

which is also strongly hyperbolic (in fact, symmetric hyperbolic), and so in this case the auxiliary constraint system is well-posed.

APPENDIX C: SYMMETRIC HYPERBOLICITY AND CHARACTERISTIC VARIABLES

Assume that P^i admits a symmetriser H . As H is Hermitian and positive definite, there is an invertible matrix S such that

$$H = S^\dagger S. \quad (\text{C1})$$

From this and

$$HP^i = P^{i\dagger}H \quad (\text{C2})$$

it follows that SP^nS^{-1} is Hermitian, for any direction n_i . Therefore it can be diagonalised by an orthogonal matrix R (which generally depends on n_i), or

$$SP^nS^{-1} = R^{-1}\Lambda R, \quad (\text{C3})$$

where Λ is diagonal. Therefore

$$P^n = T^{-1}\Lambda T, \quad T \equiv RS, \quad (\text{C4})$$

and we have proved that symmetric hyperbolicity implies strong hyperbolicity.

Furthermore, as R is orthogonal,

$$H = S^\dagger(R^\dagger R)S = T^\dagger T, \quad (\text{C5})$$

and so there is a preferred basis, namely the rows of T (which generally depends on n_i), of left eigenvectors of P^n in which H is the unit matrix. In terms of the original basis

$$H = T^\dagger T, \quad HP^n = T^\dagger \Lambda T. \quad (\text{C6})$$

In quadratic forms the same fact can be written as

$$\epsilon = \sum_\alpha \mathbf{u}_\alpha^2, \quad \phi^n = \sum_\alpha \lambda_\alpha \mathbf{u}_\alpha^2. \quad (\text{C7})$$

where the sum is over the characteristic variables in the basis encoded in the rows of T .

APPENDIX D: THE KWB FORMULATION OF THE MAXWELL EQUATIONS

We use this formulation of the Maxwell equations to illustrate some of the differences between the reduction approach and the second-order approach, namely the existence of “short” and “full” characteristic variables, the split of the free parameters of H into those that are reduction parameters and those that are not, and the relation between the characteristic variables of the main and constraint systems. We work initially in the first-order reduction approach, and then re-interpret the same calculations in the language of the second-order approach afterwards.

The system has been discussed in [10, 14]. In flat spacetime, in radiation gauge and in the absence of charges (source terms), it is

$$\dot{A}_i = -E_i, \quad (\text{D1})$$

$$\dot{E}_i = -A_{i,j}{}^{,j} + (1-a)A_{j,i}{}^{,j} + a\Gamma_{,i}, \quad (\text{D2})$$

$$\dot{\Gamma} = (b-1)E^i{}_{,i} \quad (\text{D3})$$

where $v = \{A_i\}$, $w = \{E_i, \Gamma\}$ are the dynamical variables, and a and b are constants that parameterise addition of the (“original”) constraints

$$C_\Gamma \equiv \Gamma - A_i{}^{,i} = 0, \quad (\text{D4})$$

$$C_E \equiv E_i{}^{,i} = 0 \quad (\text{D5})$$

to the evolution equations. Repeated indices have been raised with the metric δ^{ij} and are summed over. We shall also use δ_{ij} to decompose tensors into their normal and transverse parts. With $d_{ij} \equiv A_{j,i}$, we consider the first-order reduction.

$$\dot{A}_i = -E_i, \quad (D6)$$

$$\begin{aligned} \dot{E}_i &= -d_{ji}{}^{,j} + a\Gamma_{,i} + (1-a)d_j{}^{,j}{}_{,i} \\ &\quad + c(d_{ij}{}^{,j} - d_j{}^{,j}{}_{,i}), \end{aligned} \quad (D7)$$

$$\dot{\Gamma} = (b-1)E^i{}_{,i}, \quad (D8)$$

$$\dot{d}_{ij} = -E_{j,i}. \quad (D9)$$

The constant c parameterises B_3^{ij} , and $D_i{}^{jk}$ has been set to zero. For the other reduction parameters we have made the standard choice (29), so that the A_i decouple from the E_i and Γ .

The matrix \mathcal{P} , leaving out the zero rows and columns corresponding to A_i , is block-diagonal with the blocks

$$\begin{pmatrix} 0 & a & -a & 1-a-c \\ b-1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_n \\ \Gamma \\ d_{nn} \\ d_{qq} \end{pmatrix}, \quad (D10)$$

$$\begin{pmatrix} 0 & -1 & c \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_A \\ d_{nA} \\ d_{An} \end{pmatrix}, \quad (D11)$$

$$\begin{pmatrix} 0 \\ \hat{d}_{AB} \end{pmatrix}. \quad (D12)$$

Here $d_{qq} \equiv q^{ij}d_{ij}$, and \hat{d}_{AB} represents the transverse trace-free object $q_i^k q_j^l d_{kl} - (1/2)q_{ij}d_{qq}$. The characteristic variables are

$$\mathbf{u}_0 \equiv \Gamma + (b-1)d_{nn}, \quad (D13)$$

$$\mathbf{u}_{\pm} \equiv a(\Gamma - d_{nn}) + (1-a-c)d_{qq} \pm \sqrt{ab}E_n, \quad (D14)$$

$$\mathbf{u}_{\pm A} \equiv d_{nA} - cd_{An} \mp E_A, \quad (D15)$$

with speeds $(0, \pm\sqrt{ab}, \pm 1)$, and d_{qq} , d_{An} and \hat{d}_{AB} with zero speed.

The first-order reduction admits the conserved energy

$$\begin{aligned} \epsilon &= E_i E^i + d_{ij} d^{ij} - 2a\Gamma d_i{}^i + (2a-1-ab)(d_i{}^i)^2 \\ &\quad + c_1 [\Gamma + (b-1)d_i{}^i]^2 + c[(d_i{}^i)^2 - d_{ij} d^{ij}] \end{aligned} \quad (D16)$$

with the flux

$$\begin{aligned} \phi^i &= 2[a(\Gamma - d_j{}^j)E^i + d_j{}^j E^i - d^{ij}E_j] \\ &\quad + 2c(d^{jj}E_j - d_j{}^j E^i), \end{aligned} \quad (D17)$$

where c_1 is a free parameter in the energy, and c is the reduction parameter introduced above.

We now review how one would deal with the same system in a direct second-order approach [pointing out the relation to the first-order approach in *square brackets*]. In

order to show strong hyperbolicity, one would diagonalise the matrix \mathcal{A} . It is block-diagonal with the blocks

$$\begin{pmatrix} 0 & a & -a \\ b-1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_n \\ \Gamma \\ A_{n,n} \end{pmatrix} \quad (D18)$$

and

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} E_A \\ A_{A,n} \end{pmatrix}. \quad (D19)$$

[\mathcal{A} is the sub-matrix of \mathcal{P} obtained by suppressing the rows and columns relating to d_{Ai} . In a different point of view, we could set the rows and columns relating to d_{Ai} in \mathcal{P} to zero by allowing the reduction parameters to depend explicitly on n_i . In our example, this corresponds to setting $c = 1 - a$ in (D10) but $c = 0$ in (D11).]

The “short” characteristic variables of the second-order system are obtained as eigenvectors of \mathcal{A}^\dagger . They are

$$\mathbf{u}'_0 \equiv \Gamma + (b-1)A_{n,n}, \quad (D20)$$

$$\mathbf{u}'_{\pm} \equiv a(\Gamma - A_{n,n}) \pm \sqrt{ab}E_n, \quad (D21)$$

$$\mathbf{u}'_{\pm A} \equiv A_{A,n} \mp E_A, \quad (D22)$$

with speeds $(0, \pm\sqrt{ab}, \pm 1)$. [These are the characteristic variables of the reduction, minus all terms in $d_{Ai} = A_{i,A}$.]

To show symmetric hyperbolicity of the second-order system, we find an energy. The most general one is

$$\begin{aligned} \epsilon &= E_i E^i + A_{i,j} A^{i,j} - 2a\Gamma A_i{}^i + (2a-1-ab)(A_i{}^i)^2 \\ &\quad + c_1 [\Gamma + (b-1)A_i{}^i]^2 + c_2 [(A_i{}^i)^2 - A_{i,j} A^{j,i}] \end{aligned} \quad (D23)$$

with the flux

$$\begin{aligned} \phi^i &= 2[a(\Gamma - A_j{}^j)E^i + A_j{}^j E^i - A^{ij}E_j] \\ &\quad + 2c_2(A^{ij}E_j - A_j{}^j E^i), \end{aligned} \quad (D24)$$

where c_1 and c_2 are free parameters. [This is identical to the energy of the reduction, except that c_2 is now not a reduction parameter. Rather, the term it multiplies is independently conserved if we allow commutation of partial derivatives.]

In order to impose maximally dissipative boundary conditions, one needs the full characteristic variables \mathbf{u}_α . We find this by expressing ϵ as a quadratic form in \mathbf{u}_α . Comparing ϵ with the \mathbf{u}'_α suggests that

$$\begin{aligned} \epsilon &= \frac{1}{2ab}(\mathbf{u}_+^2 + \mathbf{u}_-^2) + \frac{1}{2}(\mathbf{u}_{+A}\mathbf{u}^{+A} + \mathbf{u}_{-A}\mathbf{u}^{-A}) \\ &\quad + \left(c_1 - \frac{a}{b}\right)\mathbf{u}_0^2 - c_2(\mathbf{u}_{+A} + \mathbf{u}_{-A})A^{n,A} \\ &\quad + \text{quadratic in } A_{i,A}, \end{aligned} \quad (D25)$$

with $\mathbf{u}_\alpha = \mathbf{u}'_\alpha + \text{multiples of the } A_{i,A}$. [The result is equivalent to the \mathbf{u}_α of the reduction, with $d_{ij} \rightarrow A_{j,i}$ and $c \rightarrow c_2$.]

Finally, the constraint evolution system is

$$\dot{C}_\Gamma = bC_E, \quad (\text{D26})$$

$$\dot{C}_E = aC_{\Gamma,i}{}^i. \quad (\text{D27})$$

Its non-trivial characteristic variables are

$$\mathbf{c}_\pm = a\partial_n C_\Gamma \pm \sqrt{ab}C_E = \partial_n \mathbf{u}_\pm + \text{transv. deriv.}, \quad (\text{D28})$$

and these expressions hold for both the second-order system and the reduction.

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